Branching Random Walks on Trees and Hyperbolic Spaces

Longmin Wang (Nankai University)

Joint with Matthieu Dussaule, Shuwen Lai, Vladas Sidoravicius, Kai-Nan Xiang, Wenyuan Yang

17th Workshop on Markov Processes and Related Fields

BNU @ Zhuhai, November 27, 2022

Branching Markov Chains

Suppose

- $(p_k)_{k\geq 0}$ is a non-degenerate probability distribution on $\{0, 1, 2, \dots\}$,
- (Z_n) is a Markov chain on X with transition probabilities $p(x, y)$.

A branching Markov chain, or branching random walk (BRW) is a particle system defined inductively:

- At time 0, we have one particle at $x \in X$;
- Between time n and $n+1$, the process performs two steps: branching and movement.
  - Each particle produces independently descendants according to $(p_k)_{k\geq 0}$ and dies;
  - Each of these descendants moves one step independently according to the Markov chain (Z_n) on X.

Phase Transition

- Let $r := \sum_{k=0}^{\infty} k p_k$ be the mean offspring of the Branching Markov chain,
- Let $R^{-1} := \limsup_{n\to\infty} \left[p_n(x,\, y)\right]^{1/n}$ with $p_n(x,\, y)$ the n-step transition probability of the Markov chain.

Theorem (Benjamini-Peres '94, Gantert-Müller '06)

- *If $r \le 1$, then the BRW is extinct;*
- *If $r > R$, then the BRW is recurrent on the set of non-extinction;*
- *If $1 < r \le R$, then the BRW is transient, that is, it eventually leaves any finite subset of X.*

Intuition: If $r < R$, then

$$\mathbf{E}\left[\sharp \text{ visits to } x\right] = \sum_{n=0}^{\infty} r^n p_n(e,\, x) \le \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n < \infty.$$

Question

Denote by $\mathcal{P}$ the trace of the BRW. Then

- if $r \leq 1$, then $\mathcal{P}$ is finite;
- if $r > R$, then $\mathcal{P} = X$;
- if $1 < r \leq R$, then $\mathcal{P}$ is a random subset of X.

Question

Suppose $1 < r \leq R$.

- *Given a boundary of X, what is the Hausdorff dimension of the limit set $\Lambda(r) = \overline{\mathcal{P}} \setminus X$?*

Homogeneous Tree

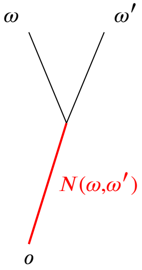
Assume $\mathbb{T}^m$ with $m \geq 3$ is the homogeneous tree with boundary

$$\partial \mathbb{T}^m = \{\text{all semi-infinite rays from } o\}.$$

For $\omega,\ \omega' \in \partial \mathbb{T}^m$, let $N(\omega,\ \omega')$ be the length of the common segment of ω and ω'. Then

$$d_{\mathrm{e}}(\omega,\ \omega') = \mathrm{e}^{-N(\omega,\omega')}$$

defines a metric on $\partial \mathbb{T}^m$.



Under d_{e}, the Hausdorff dimension $\mathrm{Hdim}\, \partial \mathbb{T}^m = \log(m-1)$.

Literature

Theorem (Liggett '96, Hueter-Lalley '00)

Let $\Lambda(r)$ be the limit set of a symmetric, nearest-neighbor branching random walk with mean offspring r on $\mathbb{T}^m$ and let $h(r) = \mathrm{Hdim}\,\Lambda(r)$. Then

- *For $1 \le r \le R$, $h(r)$ is continuous, strictly increasing and nonrandom;*
- *$h(R) \le \frac{1}{2}\,\mathrm{Hdim}\,\partial\mathbb{T}^m$; and the equality holds iff the BRW is simple.*
- *$h(r)$ has critical exponent $\frac{1}{2}$ at R: there is a constant $C > 0$ such that*

$$h(R) - h(r) \sim C\,(R - r)^{\frac{1}{2}}, \quad r \uparrow R.$$

- Branching Brownian motion on hyperbolic spaces: Lalley-Sellke '97, Karpelevich-Pechersky-Suhov '98
- BRW on free products of groups: Candellero-Gilch-Müller '12

Random Walks on Hyperbolic Spaces

- Let X be a closed convex subset of the real hyperbolic space $\mathbb{H}^d$, and
- Γ a discrete group of isometries of $\mathbb{H}^d$ acting geometrically on X.

Assume that

- μ is a symmetric probability measure on Γ with finite support. Here we say μ is symmetric if $\mu(g^{-1}) = \mu(g)$ for all $g \in \Gamma$.
- μ is admissible (i.e. the support of μ generates Γ as a semigroup).

The random walk (Z_n) on Γ is the Markov chain with transition probabilities $p(x, y) = \mu\left(x^{-1}y\right)$, starting at the neutral element e.

Fix $o \in X$. We call $(Z_n o)$ the random walk on X.

BRW on Hyperbolic Spaces

Recall that R is the radius of convergence for $G_r(x, y) = \sum_{n=0}^{\infty} r^n p_n(x, y)$.

Consider a BRW on X with mean offspring r and let $\mathcal{P}$ be its trace. The limit set $\Lambda(r)$ of the BRW is the set of accumulation points of $\mathcal{P}$ in ∂X.

Theorem (Sidoravicius-W.-Xiang '20+, Dussaule-W.-Yang '22+)

For $1 < r \le R$,

- *there is* $\kappa < +\infty$ *such that almost surely for every* $\xi \in \Lambda(r)$, *as* $x \to \xi$ *along the geodesic* $[o, \xi]$,

$$\limsup \frac{d(x, \mathcal{P}o)}{\log d(o, x)} < \kappa.$$

- $$\operatorname{Hdim} \Lambda(r) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{g:\ n \le d(o, go) < n+1} G_r(e, g)$$

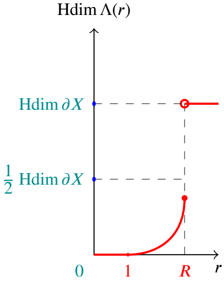
Gap at R

By the Cauchy-Schwarz inequality,

$$\sum_{n \le d(o,go) < n+1} G_r(e,\, g) \le \left[\sum_{n \le d(o,go) < n+1} 1 \right]^{\frac{1}{2}} \left[\sum_{n \le d(o,go) < n+1} G_r(e,\, g)^2 \right]^{\frac{1}{2}}.$$

- By Gouëzel '14, $\sum_{n \le d(o,go) < n+1} G_r(e,\, g)^2 \le C$.
- By Coornaert '93,

$$\operatorname{Hdim} \partial X = \lim_{n \to \infty} \frac{1}{n} \log \sharp \{ g :\, n \le d(o,\, go) < n+1 \}.$$



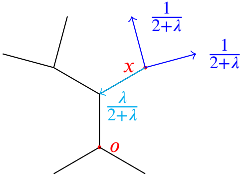
Thus

$$\operatorname{Hdim} \Lambda(r) \le \frac{1}{2} \operatorname{Hdim} \partial X.$$

Biased Random Walks

For $\lambda > 0$, the λ-biased random walk on the homogeneous tree $\mathbb{T}^{d+1}$ with root o is the Markov chain with transition probabilities

$$p(x, y) = \begin{cases} \frac{1}{d+1}, & \text{if } y^- = x = o, \\ \frac{1}{d+\lambda}, & \text{if } y^- = x,\ x \neq o, \\ \frac{\lambda}{d+\lambda}, & \text{if } x^- = y, \end{cases}$$



where x^- is the parent of x. Then the radius of convergence is

$$R = \begin{cases} \frac{d+\lambda}{2\sqrt{d\lambda}}, & \text{if } 0 < \lambda < d, \\ 1, & \text{if } \lambda \geq d. \end{cases}$$

We have for $0 < \lambda < d$,

$$h(r) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{d(o,x)=n} G_r(o, x) = \log \left[\frac{d+\lambda - \sqrt{(d+\lambda)^2 - 4d\lambda r^2}}{2\lambda r} \right].$$

Biased BRW

Consider the λ-biased BRW on $\mathbb{T}^{d+1}$ with mean offspring r.

- If $d^{-1} < \lambda < d$, then for $1 < r \le R$, $\mathrm{Hdim}\,\Lambda(r) = h(r)$. In particular,

$$\mathrm{Hdim}\,\Lambda(R) = \frac{1}{2}\left[\log d - \log \lambda\right] < \log d.$$

- If $\lambda = d^{-1}$, then for $1 < r \le R$, $\mathrm{Hdim}\,\Lambda(r) = h(r)$. In particular, $\mathrm{Hdim}\,\Lambda(R) = \log d$, and $\mathrm{Hdim}\,\Lambda(r)$ is continuous for $r > 0$.
- If $0 < \lambda < d^{-1}$, then with $R_0 = \frac{d+\lambda}{d\lambda+1}$,

$$\mathrm{Hdim}\,\Lambda(r) = \begin{cases} h(r), & \text{if } 1 < r < R_0, \\ \log d, & \text{if } R_0 \le r \le R. \end{cases}$$

  In particular, $\mathrm{Hdim}\,\Lambda(r)$ is continuous for $r > 0$.

Random Walks on Trees

Let T be a locally finite, infinite tree with root o. For $x \in T$, the cone C_x is the subtree rooted at x. Denote

$$\mathcal{C} = \left\{ \text{isomorphism classes of } C_x : \ x \in T, \ x \neq o \right\}.$$

For $x \in T \setminus \{o\}$, define $\tau(x) \in \mathcal{C}$ to be the type of the cone C_x. For i, $j \in \mathcal{C}$, we assign $\mathrm{d}(i, j)$ directed edges from i to j, where $\mathrm{d}(i, j)$ is the number of type-j children of x with $\tau(x) = i$.
We say T has finitely many cone types, if $\mathcal{C}$ is irreducible and finite.

Consider a random walk on T with transition probabilities $p(x, y)$. Assume

$$\tau(x) = i, \, \tau(y) = j \text{ with } y \in C_x \quad \Longrightarrow \quad p(x, y) = \mathrm{p}(i, j),$$

where $\mathrm{p}(i, j)$ is a matrix on $\mathcal{C}$. Then

$$p(x, x^-) = 1 - \sum_j \mathrm{d}(i, j)\mathrm{p}(i, j).$$

See Lyons '90, Takacs '97, Nagnibeda-Woess '02.

BRW on Trees

Recall that

$$h(r) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{d(o,x)=n} G_r(o, x).$$

Define

$$R_0 = \sup\left\{ r > 0 : G_r(o, x) \le 1 \quad \text{for all large } x \right\}.$$

Theorem (Lai-W. in progress)

Let $\Lambda(r)$ be the limit set of the BRW with mean offspring r on trees with finitely many cone types. Then for $1 < r \le R_0$,

$$\mathrm{Hdim}\, \Lambda(r) = h(r).$$

Remark If the group $\mathrm{AUT}(T, P)$ acting on (T, P) transitively, then $R_0 = R$.

Problems

1. On trees with finitely many cone types, find $\mathrm{Hdim}\,\Lambda(r)$ for $R_0 < r \leq R$.
2. On hyperbolic spaces, find conditions for $\mathrm{Hdim}\,\Lambda(R) = \frac{1}{2}\,\mathrm{Hdim}\,\partial X$.
   - Fundamental inequality: $h \leq l\,v$ (c.f. Guivarc'h '79) Blachère-Haïssinsky-Mathieu '11, Gouëzel-Mathéus-Maucourant '18, Dussaule-Gekhtman '20: For random walks on (relatively) hyperbolic groups Γ, $h = l\,v$ only if Γ is virtually free.
   - $\sum_{n\leq d(o,go)<n+1} G_R(e,g) \asymp \left[\sum_{n\leq d(o,go)<n+1} 1\right]^{\frac{1}{2}} \left[\sum_{n\leq d(o,go)<n+1} G_R(e,g)^2\right]^{\frac{1}{2}}$.
3. Percolation, Contact processes, ...

Thank You for Your Attention!